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Higher-order polarization on the Poincaré group and the position operator*

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Abstract. The quantization of the free relativistic spinning particle is revised on the basis of a group approach to quantization. In momentum space, the wavefunctions provide the minimal canonical representation of mass m and spin j of the Poincaré group $\mathcal{P}_+^\uparrow$. The quantization in configuration space requires, as in many other physical systems, polarizations of higher-order type. This higher-order polarization technique turns out to be a natural framework for studying localizability and to provide a position operator.

1. Introduction

Some years ago a considerably detailed analysis of the quantum dynamics of the (spinless) free relativistic particle was developed in [1]. The aim of that paper was to clarify the group-theoretical structure of this elementary system. The starting point of such a study was clearly the 10-parameter Poincaré group, or, more precisely, a pseudo-extension of it. The wavefunctions provided the (spin-zero) irreducible representations of the Poincaré group in momentum space.

The framework used in [1] is a group approach to quantization (GAQ) [2, 3], a formalism in which a Lie group (or supergroup) law is the only input of the theory. Essentially, GAQ is a definite prescription to go from a group law, which is intended to be the 'relevant' symmetry of a given physical system, to its unitary irreducible representations in a Hilbert space made up only of functions on the group manifold, thus achieving the complete solution to the quantum-mechanics problem. More precisely, GAQ starts with a Lie group $\widetilde{G}$ containing a preferred U(1) subgroup as for the case of a U(1) central extension [4] (or pseudo-extension). This U(1) subgroup accounts for the usual phase invariance of quantum mechanics. The mere existence of this subgroup allows us to distinguish between two types of variables in the group: those whose corresponding generators produce a central term on the right-hand side of their commutator, and the remainder.The former corresponds to the set of canonically conjugate pairs of coordinates and momenta, and the latter to variables playing a role similar to that of time. Time and rotations are examples of this second type of (non-dynamical)

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variables for the spinless free particle. From a mathematical point of view we can say that the group $\tilde{G}$ is a principal bundle [5] with structure group (or fibre) U(1). If $\tilde{G}$ is a central extension of G , the kernel of the Lie algebra co-cycle [6] $\Sigma : \mathcal{G} \times \mathcal{G} \rightarrow \Re$, which is known as the characteristic subalgebra $\mathcal{G}_\Theta$, generates the subgroup of non-dynamical variables.

GAQ then continues taking complex function Ψ on $\tilde{G}$ that are homogeneous of degree one in $\zeta \in U(1)$ (equivariance condition). This means, in differential form, that $\Xi\Psi = i\Psi$, where Ξ is the U(1) generator. The Hilbert space is now obtained by imposing the so-called full polarization condition, i. e., a maximal restriction that can be imposed on functions Ψ so as to fully reduce the quantum representation. A full polarization $\mathcal{P}$ is a maximal subalgebra of left-invariant vector fields $\tilde{X}^L$ containing $\mathcal{G}_\Theta$ and excluding Ξ, and the wavefunctions must be annihilated by $\tilde{X}^L$ in $\mathcal{P}$. This definition generalizes the analogous concept in ordinary geometric quantization [7–10] in that our $\mathcal{P}$ contains the non-symplectic variables associated with $\mathcal{G}_\Theta$. Since right-invariant vector fields commute with left ones, the former leave the Hilbert space invariant and achieve the quantum representation. The left-invariant one-form Θ dual to Ξ also generalizes the quantization form of geometric quantization ($d\Theta$ is no longer a symplectic form and its kernel, or more precisely its characteristic module, $\mathrm{Ker}\, d\Theta \cap \mathrm{Ker}\, \Theta$, is generated by a basis of $\mathcal{G}_\Theta$).

This paper is devoted to the study of the spin under GAQ to account for a group-theoretic formulation of relativistic quantum mechanics without any reference to fermionic variables. We study first, in section 2, the non-relativistic case by introducing a pseudo-extension of the SU(2) subgroup (SU(2) is semi-simple and therefore all the central extensions are trivial) of the Galilei group parametrized by a (necessarily) half-integer j. The objective of this extension is to take two SU(2) generators out of the characteristic subalgebra and to convert the corresponding parameters to a canonically conjugated pair of a coordinate and a momentum which provides the spin degrees of freedom. We have included an appendix devoted to the general study of the SU(2) representations on the basis of GAQ.

The introduction of a SU(2) pseudo-co-cycle on the Poincaré group is carried out in section 3 without any special difficulty. The only difference with respect to the Galilean situation is the complexity of the group law and the fact that now a central term appears on the right-hand side of a commutator between two boosts. We find that the resulting quantum representation is the minimal canonical representation of mass m and spin j of the Poincaré group $\mathcal{P}_+^\uparrow$ [11–13] in momentum space (we discard the discrete elements for the sake of simplicity and consider only a single orbit). Although the wavefunctions factor out in orbital and spin parts, as in the non-relativistic case, the operators mix both parts properly.

The quantization of the relativistic spinning particle in configuration space requires, as in many other physical systems, the introduction of the concept of higher-order polarization, a concept originally introduced to solve anomaly problems in both finite- [14] and infinite-dimensional quantum systems [15].

The anomaly problem can be viewed in any geometric approach to quantization (this is an unsolvable problem in geometric quantization) as the absence of a first-order (full) polarization [14]. In that case, the definition of full polarization in the GAQ can be generalized so as to include operators in the left-enveloping algebra of the original group. Higher-order polarizations may also be introduced to quantize a given non-anomalous physical system in a representation different from the one provided by the existing first-order full polarization. This is precisely the way in which the standard second-order Schrödinger equation (in configuration space) appears in non-relativistic quantum mechanics.

This paper is organized as follows. In section 2 we quantize the non-relativistic spinning particle, in both momentum and configuration space, as an example that shows how the

GAQ including higher-order polarizations is applied. Section 3 is devoted to the study of the relativistic spinning particle. We first obtain the minimal, canonical representation of mass m and spin j of the Poincaré group in momentum space. Then we introduce a higher-order polarization characterizing the quantities related to the configuration-space representation and provide explicit solutions to the pseudo-differential polarization (and motion) equations. Finally, we consider the classical solution manifold and Noether invariants as classical counterparts of the previously introduced operators. In section 4 we comment on several particularities of our higher-order polarization mechanism. An appendix revises the irreducible representation of the SU(2) group according to our group-theoretic methods.

2. The free non-relativistic spinning particle

The quantization of the free non-relativistic particle in momentum space by means of GAQ was already considered in the first paper where this formalism was introduced [2] (see also [3]). The starting point was of course the eleven-parameter group, consisting of a one-dimensional central extension of the Galilei group parametrized by the mass. The rotation subgroup there played the mere role of a spectator in the sense that none of its variables were dynamical in character. The characteristic subgroup was composed of rotations as well as time translations [3] (in [2] rotations were not even considered). Nevertheless, as is thoroughly discussed in the appendix, although the subgroup SU(2) does not have non-trivial central extensions, some co-boundaries allow us to extend this subgroup in a way that resembles a non-trivial extension parametrized by a half-integer j (see [16] for the analogous case of SL(2,R)). Thus, two out of the three parameters of SU(2) can be considered as being the canonically conjugated coordinates that account for the spin of the free particle.

Let us consider, therefore, a group law for the extended Galilei group with a central extension provided by a non-trivial co-cycle, parametrized by the mass m, as well as a pseudo-cocycle (in reality a co-boundary) parametrized by the spin j. This co-boundary is generated by a local linear function on the parameter θ of the Cartan subgroup U(1) $\in$ SU(2) (not to be confused with the central U(1) subgroup). In this section, and for the sake of simplicity, we shall fix the spin quantization axis (see appendix) in the z direction, i.e. $\boldsymbol{n} = (0, 0, 1)$. The group law is:

$$
\begin{aligned}
t'' &= t' + t \\
\boldsymbol{x}'' &= \boldsymbol{x}' + R'(\boldsymbol{\epsilon})\,\boldsymbol{x} + \boldsymbol{v}'t \\
\boldsymbol{v}'' &= \boldsymbol{v}' + R'(\boldsymbol{\epsilon})\,\boldsymbol{v} \\
\boldsymbol{\epsilon}'' &= \sqrt{1-\boldsymbol{\epsilon}'^2/4}\,\boldsymbol{\epsilon} + \sqrt{1-\boldsymbol{\epsilon}^2/4}\,\boldsymbol{\epsilon}' + \frac{1}{2}\boldsymbol{\epsilon}' \times \boldsymbol{\epsilon} \\
\zeta'' &= \zeta'\zeta\, \mathrm{e}^{im[\boldsymbol{x}'R'\boldsymbol{v}+t(\boldsymbol{v}'R'\boldsymbol{v}+v^2/2)]/\hbar}\, \mathrm{e}^{2ij(\theta''-\theta'-\theta)}.
\end{aligned}
\tag{1}
$$

The rotations $R(\boldsymbol{\epsilon})$ are given in (A24).

The left-invariant and right-invariant vector fields are, respectively:

$$
\begin{aligned}
\tilde{X}^L_{(t)} &= \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{1}{2}mv^2\,\Xi \qquad \tilde{X}^L_{(\boldsymbol{x})} = R\frac{\partial}{\partial \boldsymbol{x}} \\
\tilde{X}^L_{(\boldsymbol{v})} &= R\left(\frac{\partial}{\partial \boldsymbol{v}} + m\boldsymbol{x}\,\Xi\right)
\end{aligned}
\tag{2}
$$

$$\tilde{X}^L_{(\epsilon)} = \tilde{X}^{L\ \mathrm{SU(2)}}_{(\epsilon)} \qquad \tilde{X}^L_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi$$

and

$$\begin{aligned}
&\tilde{X}^R_{(t)} = \frac{\partial}{\partial t} \qquad \tilde{X}^R_{(\boldsymbol{x})} = \frac{\partial}{\partial \boldsymbol{x}} + m\boldsymbol{v}\,\Xi \\
&\tilde{X}^R_{(\boldsymbol{v})} = \frac{\partial}{\partial \boldsymbol{v}} + t\frac{\partial}{\partial \boldsymbol{x}} + m\boldsymbol{v}t\,\Xi \\
&\tilde{X}^R_{(\epsilon)} = \tilde{X}^{R\ \mathrm{SU(2)}}_{(\epsilon)} + \boldsymbol{x}\times\frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{v}\times\frac{\partial}{\partial \boldsymbol{v}} \\
&\tilde{X}^R_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi.
\end{aligned} \tag{3}$$

From (2) the quantization form we can derive

$$\Theta = -\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{p} - \frac{p^2}{2m}\mathrm{d}t + \Theta^{\mathrm{SU(2)}} + \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{4}$$

where $\Theta^{\mathrm{SU(2)}}$ has been given in (A26).

Unlike the case for spin zero [3], where all the rotation generators were included in the characteristic subalgebra $\mathcal{G}_\Theta$, here for $j \neq 0$ we have only

$$\mathcal{G}_\Theta = \langle \tilde{X}^L_{(t)}, \tilde{X}^L_{(\epsilon^3)} \rangle \tag{5}$$

thus indicating that here the variables ϵ_1 and ϵ_2 will play the role of a coordinate–momentum pair (in connection with this, see the Poisson structure in (A27),(A29), which could be translated here by adding the spatial part of the Noether invariants: this will be done later for the Poincaré case).

We shall take the following full polarization for the extended Galilei group with spin:

$$\mathcal{P} = \langle \tilde{X}^L_{(t)}, \tilde{X}^L_{(\boldsymbol{x})}, \tilde{X}^L_{(\epsilon^3)}, \tilde{X}^L_{(\epsilon^2)} + \mathrm{i}\tilde{X}^L_{(\epsilon^1)} \rangle. \tag{6}$$

The polarization conditions $\tilde{X}^L\ \Psi = 0, \forall \tilde{X}^L \in \mathcal{P}$, when acting on functions which satisfy the condition $\Xi\Psi = \mathrm{i}\Psi$, that is, $\Psi(\zeta, \boldsymbol{x}, \boldsymbol{v}, t, \epsilon) = \zeta\Psi(\boldsymbol{x}, \boldsymbol{v}, t, \epsilon)$, give the following wavefunctions for the free non-relativistic particle with spin:

$$\Psi^{(j)}_\mu = \zeta \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}mv^2t}\chi^{(j)}_\mu(\epsilon)\varphi(\boldsymbol{v}) \tag{7}$$

where $\chi^{(j)}_\mu$ is given in (A13), and it is assumed that the change (A18) to coordinates ϵ is made in $\chi^{(j)}_\mu$. The operators $\tilde{X}^R_{(\epsilon,\zeta)}$ acting on the functions (7) give a representation of SU(2)$\tilde{\otimes}$U(1). The most common operators $\hat{J}_0$, $\hat{J}_+$ and $\hat{J}_-$ can be obtained from $\tilde{X}^R_{(\epsilon)}$, by using the relationships (A14) and (A18). Clearly the action of the $\hat{J}$ operators on the j and μ indices of the Galilei wavefunctions is the same as in (A15).

The globality argument given after (A15) translates here as a whole; in particular, j in (7) has to be a half-integer and is identified as the spin of the particle. The subindex μ ranges from $-j$ to j.

In order to compare these Galilean results with the relativistic ones, let us see the explicit form of the action of the physical operators $\hat{E}^{\mathrm{nr}} = \mathrm{i}\hbar\tilde{X}^R_{(t)}$, $\hat{\boldsymbol{P}}^{\mathrm{nr}} = -\mathrm{i}\hbar\tilde{X}^R_{(\boldsymbol{x})}$, $\hat{\boldsymbol{K}}^{\mathrm{nr}} = (\mathrm{i}\hbar/m)\tilde{X}^R_{(\boldsymbol{v})}$ and $\hat{\boldsymbol{J}}^{\mathrm{nr}} = -\mathrm{i}\hbar(\tilde{X}^R_{(\epsilon)} + j\boldsymbol{n}\Xi)$ (where superscript 'nr' stands for 'non-relativistic'):

$$\begin{aligned}
&\hat{E}^{\mathrm{nr}}\varphi(\boldsymbol{v}) = \tfrac{1}{2}mv^2\varphi(\boldsymbol{v}) \qquad \hat{\boldsymbol{P}}^{\mathrm{nr}}\varphi(\boldsymbol{v}) = m\boldsymbol{v}\varphi(\boldsymbol{v}) \\
&\hat{\boldsymbol{K}}^{\mathrm{nr}}\varphi(\boldsymbol{v}) = \mathrm{i}\frac{\hbar}{m}\frac{\partial}{\partial \boldsymbol{v}}\varphi(\boldsymbol{v})
\end{aligned} \tag{8}$$

and

$$\hat{\mathcal{J}}^{\rm nr}\varphi(\boldsymbol{v})\chi_\mu^{(j)}(\epsilon) = \left[-i\hbar \boldsymbol{v}\times\frac{\partial}{\partial \boldsymbol{v}} - i\hbar\left(\tilde{X}^R_{(\epsilon)} + ij\boldsymbol{n}\right)\right]\varphi(\boldsymbol{v})\chi_\mu^{(j)}(\epsilon). \tag{9}$$

Before considering the quantization in configuration space we must comment that since [1] was published, the concept of polarization has been generalized, mainly due to the fact that richer (more general) ideas were needed to deal with the problem of anomalous systems. These ideas have been explained in detail in [14], so that here we intend to relate briefly what is relevant for the cases examined in this paper.

The characteristic subalgebra usually includes all variables which are not part of any coordinate–momentum pair of the dynamical system. The polarization subalgebra will then consist of the characteristic subalgebra and half of the rest of the variables, so to speak. In this case, we say we have a full polarization. In retrospect we can say that all the polarizations used in [1] were full polarizations. It may occur, however, that some variables which are not included in any coordinate–momentum pair, are not included in the characteristic subalgebra either. This type of structure, i.e. non-full polarization or just polarization (some examples have been given in [14]), would lead to representations which are not fully reduced, and thus an improved technique of quantization has to be employed. These ideas have been used to characterize and solve anomalous systems, although they can also be applied to non-anomalous systems if we want them to appear in a different 'representation'. For instance, there is no (first-order) full polarization that permits the quantization of the free particle in the x-representation. This 'representation' would require $\tilde{X}^L_{(\boldsymbol{v})}$ to be in the polarization subalgebra, but then $\tilde{X}^L_{(t)}$ would have to be excluded since $[\tilde{X}^L_{(t)}, \tilde{X}^L_{(\boldsymbol{v})}] \sim \tilde{X}^L_{(\boldsymbol{x})}$.

As mentioned in the introduction, it is possible to generalize the notion of (first-order) polarization by allowing the operators in the left-enveloping algebra to belong to the polarization. Thus, we define a higher-order polarization as a maximal subalgebra of the left-enveloping algebra of G excluding the central generator and containing a first-order polarization.

Starting from the non-full, first-order polarization $\langle \tilde{X}^L_{(\boldsymbol{v})}, \tilde{X}^L_{(\epsilon^3)}, \tilde{X}^L_{(\epsilon^2)} + \mathrm{i}\tilde{X}^L_{(\epsilon^1)}\rangle$, we can modify $\tilde{X}^L_{(t)}$, by adding higher-order terms to it, so that it can enter the polarization subalgebra. We then obtain:

$$\mathcal{P}^{\rm HO} = \left\langle \tilde{X}^L_{(\boldsymbol{v})}, \tilde{X}^L_{(\epsilon^3)}, \tilde{X}^L_{(\epsilon^2)} + \mathrm{i}\tilde{X}^L_{(\epsilon^1)}, \tilde{X}^L_{(t)} - \frac{i\hbar}{2m}\left(\tilde{X}^L_{(\boldsymbol{x})}\right)^2\right\rangle. \tag{10}$$

The new wavefunctions are

$$\Psi_\mu^{(j)} = \zeta \mathrm{e}^{-(\mathrm{i}m/\hbar)\boldsymbol{v}\cdot\boldsymbol{x}}\chi_\mu^{(j)}(\epsilon)\Phi(\boldsymbol{x}, t) \tag{11}$$

where $\Phi(\boldsymbol{x}, t)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi. \tag{12}$$

Note that the wavefunction (11) could be rewritten as

$$\Psi_\mu^{(j)} = \zeta \mathrm{e}^{-(\mathrm{i}m/\hbar)\boldsymbol{v}\cdot\boldsymbol{x}}\mathrm{e}^{(\mathrm{i}\hbar t/2m)\nabla^2}\chi_\mu^{(j)}(\epsilon)\kappa(\boldsymbol{x}) \tag{13}$$

where we have factorized out the 'function' $\mathrm{e}^{(\mathrm{i}\hbar t/2m)\nabla^2}$. The functions $\chi_\mu^{(j)}(\epsilon)\kappa(\boldsymbol{x})$ then constitute the minimal carrier space for the irreducible representations.

3. The free relativistic spinning particle: position operator

3.1. The momentum space representation

We now face the more interesting problem of the relativistic spinning particle. Unlike the case of the Galilei group, as was discussed in the introduction of [1], the 10-parameter Poincaré group law does not admit a central extension by a true co-cycle. Nonetheless, since it allows the application of GAQ, a pseudo-extension by a co-boundary is adequate to simulate a central extension, thereby providing a solution. This has been done in the appendix for the SU(2) case. It is also the approach that was used in section 2 of [1], and the approach we shall use in this section.

To obtain a group law for the relativistic free particle that provides non-trivial representations of the rotation part of the group we shall proceed exactly as we did in the preceding section for the non-relativistic case. Thus, we shall depart from the 10-parameter Poincaré group law, pseudo-extended 'by the mass' and pseudo-extended 'by the spin'. More precisely, we shall take the Poincaré group law given in [1] (equations (A.3) and (A.4)),

$$g'' = g' * g; \quad g = (a^\mu, \epsilon, \boldsymbol{p}) \tag{14}$$

and for the central part we take

$$\zeta'' = \zeta'\zeta\, \mathrm{e}^{(\mathrm{i}m/\hbar)\xi_\eta(g',g)} \mathrm{e}^{2\mathrm{i}j(\theta''-\theta'-\theta)} \tag{15}$$

where $\xi_\eta(g', g)$ was given in [1], equation (A.8), and $\eta = \mathrm{e}^{\mathrm{i}\theta}$ is the function of ϵ that appears in (A20). Of course, θ'' should now be written using (14). From the group law in (14), (15) we derive the left- and right-invariant vector fields (note that $p^0 = \sqrt{m^2c^2 + \boldsymbol{p}^2}$):

$$\begin{aligned}
\tilde{X}^L_{(a^0)} &= \frac{p^0}{mc}\frac{\partial}{\partial a^0} + \frac{\boldsymbol{p}}{mc}\cdot\frac{\partial}{\partial \boldsymbol{a}} + (p^0 - mc)\frac{p^0}{mc}\Xi \\
\tilde{X}^L_{(\boldsymbol{a})} &= R(\epsilon)\left[\frac{\boldsymbol{p}}{mc}\frac{\partial}{\partial a^0} + \frac{\partial}{\partial \boldsymbol{a}} + \frac{1}{mc(p^0+mc)}\boldsymbol{p}\,(\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{a}}) + \frac{(p^0-mc)}{mc}\boldsymbol{p}\,\Xi\right] \\
\tilde{X}^L_{(\boldsymbol{p})} &= R(\epsilon)\left[\frac{\partial}{\partial \boldsymbol{p}} + \frac{1}{mc(p^0+mc)}\boldsymbol{p}\,(\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{p}}) + \left(\boldsymbol{a} + \frac{(\boldsymbol{a}\cdot\boldsymbol{p})\boldsymbol{p}}{mc(p^0+mc)}\right)\Xi\right] \\
&\quad + \frac{1}{mc(p^0+mc)}R^{-1}(\epsilon)\boldsymbol{p}\times\left(\tilde{X}^{L\,\mathrm{SU(2)}}_{(\epsilon)} + jn\,\Xi\right)
\end{aligned} \tag{16}$$

$$\tilde{X}^L_{(\epsilon)} = \tilde{X}^{L\,\mathrm{SU(2)}}_{(\epsilon)} \qquad \tilde{X}^L_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi$$

$$\begin{aligned}
\tilde{X}^R_{(a^0)} &= \frac{\partial}{\partial a^0} \qquad \tilde{X}^R_{(\boldsymbol{a})} = \frac{\partial}{\partial \boldsymbol{a}} + \boldsymbol{p}\,\Xi \\
\tilde{X}^R_{(\boldsymbol{p})} &= \frac{p^0}{mc}\frac{\partial}{\partial \boldsymbol{p}} - \frac{1}{mc(p^0+mc)}\boldsymbol{p}\times\left(\tilde{X}^{R\,\mathrm{SU(2)}}_{(\epsilon)} + jn\,\Xi\right) \\
&\quad + \frac{a^0}{mc}\frac{\partial}{\partial \boldsymbol{a}} + \frac{\boldsymbol{a}}{mc}\frac{\partial}{\partial a^0} + \left[\frac{a^0}{mc}\boldsymbol{p} + \frac{p^0}{mc}\boldsymbol{a} - \boldsymbol{a}\right]\Xi
\end{aligned} \tag{17}$$

$$\tilde{X}^R_{(\epsilon)} = \tilde{X}^{R\,\mathrm{SU(2)}}_{(\epsilon)} + \boldsymbol{a}\times\frac{\partial}{\partial \boldsymbol{a}} + \boldsymbol{p}\times\frac{\partial}{\partial \boldsymbol{p}} \qquad \tilde{X}^R_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi.$$

The non-zero commutation relations between right vector fields are:

$$[\tilde{X}^R_{(a^i)}, \tilde{X}^R_{(p^j)}] = \delta_{ij}(\frac{1}{mc}\tilde{X}^R_{(a^0)} - \frac{1}{\hbar}\Xi)$$

$$\begin{aligned}
[\tilde{X}^R_{(a^0)}, \tilde{X}^R_{(p^j)}] &= \frac{1}{mc}\tilde{X}^R_{(a^j)} \\
[\tilde{X}^R_{(p^i)}, \tilde{X}^R_{(p^j)}] &= \frac{1}{m^2c^2}\eta_{ij.}^{\ \ k}\left(\tilde{X}^R_{(\epsilon^k)} + jn_k\Xi\right) \\
[\tilde{X}^R_{(\epsilon^i)}, \tilde{X}^R_{(\epsilon^j)}] &= -\eta_{ij.}^{\ \ k}\left(\tilde{X}^R_{(\epsilon^k)} + jn_k\Xi\right) \\
[\tilde{X}^R_{(a^i)}, \tilde{X}^R_{(\epsilon^j)}] &= -\eta_{ij.}^{\ \ k}\tilde{X}^R_{(a^k)} \\
[\tilde{X}^R_{(p^i)}, \tilde{X}^R_{(\epsilon^j)}] &= -\eta_{ij.}^{\ \ k}\tilde{X}^R_{(p^k)}.
\end{aligned} \tag{18}$$

The quantization form obtained from (16) is

$$\Theta = -(p^0 - mc)\,\mathrm{d}a^0 - \left(\boldsymbol{a} + \frac{\boldsymbol{p}\times\boldsymbol{S}}{mc(p^0+mc)}\right)\cdot\mathrm{d}\boldsymbol{p} + \frac{(\boldsymbol{S}\times\boldsymbol{n})\cdot\mathrm{d}\boldsymbol{S}}{(j+\boldsymbol{S}\cdot\boldsymbol{n})} + \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{19}$$

The condition $i_X\mathrm{d}\Theta = 0 = i_X\Theta$ determines the characteristic subalgebra

$$\mathcal{G}_\Theta = \langle \tilde{X}^L_{(a^0)}, \boldsymbol{n}\cdot\tilde{X}^L_{(\epsilon)}\rangle \tag{20}$$

and we have a full polarization, which we take as

$$\mathcal{P} = \langle \tilde{X}^L_{(a^0)}, \tilde{X}^L_{(\boldsymbol{a})}, \tilde{X}^L_{(\epsilon)} - \mathrm{i}\boldsymbol{n}\times\tilde{X}^L_{(\epsilon)}\rangle. \tag{21}$$

The polarization conditions $\tilde{X}^L\ \Psi = 0, \forall\ \tilde{X}^L \in \mathcal{P}$ together with $\Xi\Psi = \mathrm{i}\Psi$ give the following wavefunctions:

$$\begin{aligned}
\Psi^{(j)}_\mu &= \zeta \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(p^0-mc)a^0}\Phi^{(j)}_\mu(\epsilon, \boldsymbol{p}) \\
\Phi^{(j)}_\mu &= \chi^{(j)}_\mu(\epsilon)\Phi(\boldsymbol{p})
\end{aligned} \tag{22}$$

where again $\chi^{(j)}_\mu$ are the functions given in (A13) under the change (A18) in the argument. Note that $\Phi(\boldsymbol{p})$ is the same for all $\Psi^{(j)}_\mu$, regardless of spin. It is remarkable that, until now, the quantization of the pseudo-extended Poincaré group law in (14),(15) has run completely parallel to the non-relativistic (Galilei) case that was done in section 2. Special heed should be paid to the structural similarity between the wavefunctions in (7) and (22). Indeed, consideration should be made of the fact that in the relativistic case there is some sort of mixture between boosts ($\boldsymbol{p}$) and rotations (ϵ), as is apparent from the commutation relation in the third equation in (18), a fact without a non-relativistic counterpart. It is surprising then that no mixture appears in the relativistic wavefunctions where ϵ and $\boldsymbol{p}$ have clearly factored out.

The puzzle above is solved when we evaluate the action of the physical operators on the wavefunctions, since it is on the operators that the mixture between boost and rotation parameters will take place. The results here, concerning $\tilde{X}^R_{(\boldsymbol{p})}$, will differ markedly from the third equation in (8). Let us define

$$\hat{E} = \mathrm{i}\hbar\tilde{X}^R_{(a^0)};\ \ \hat{\boldsymbol{P}} = -\mathrm{i}\hbar\tilde{X}^R_{(\boldsymbol{a})};\ \ \hat{\boldsymbol{K}} = \mathrm{i}\hbar\tilde{X}^R_{(\boldsymbol{p})};\ \ \hat{\boldsymbol{J}} = -\mathrm{i}\hbar\left(\tilde{X}^R_{(\epsilon)} + j\boldsymbol{n}\Xi\right). \tag{23}$$

The action of the other operators on the wavefunctions $\Phi^{(j)}_\mu(\epsilon, \boldsymbol{p})$ in the second equation in (22) is

$$\begin{aligned}
\hat{E}\Phi^{(j)}_\mu &= (p^0 - mc)\Phi^{(j)}_\mu \\
\hat{\boldsymbol{P}}\Phi^{(j)}_\mu &= \boldsymbol{p}\,\Phi^{(j)}_\mu
\end{aligned}$$

$$\hat{J}\Phi^{(j)} = \left[-i\hbar p \times \frac{\partial}{\partial p} + \hbar\Sigma^{(j)}\right]\Phi^{(j)} \tag{24}$$

$$\hat{K}\Phi^{(j)} = \left[i\hbar\frac{p^0}{mc}\frac{\partial}{\partial p} + \frac{\hbar}{mc(p^0+mc)}p \times \Sigma^{(j)}\right]\Phi^{(j)}$$

where

$$\Sigma^{(j)}\Phi^{(j)} = -\mathrm{i}\left[\tilde{X}^{R\ \mathrm{SU(2)}}_{(\epsilon)} + jn\Xi\right]\Phi^{(j)}. \tag{25}$$

It is now helpful to note that the energy operator $\hat{E}$ gives the energy p^0 with the rest energy mc subtracted. On the one hand, we expected this from the beginning, as we are using a pseudo-extension. On the other hand, this constant term allows the non-relativistic limit to be directly performed in the expressions we have up to now: e.g. the first equation in (24) goes to the first equation in (8) under $c \to \infty$, and so does the exponential part of the wavefunction in (22) to the exponential part in (7). For the use of spin $j = 1/2$ we can write as in the appendix $\chi^{(1/2)}_{-1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^{(1/2)}_{1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and taking into account (A30), $\hat{K}$ for instance, can be rewritten as

$$\hat{K}\,|_{j=1/2} = i\hbar\frac{p^0}{mc}\frac{\partial}{\partial p} - \frac{\hbar}{mc(p^0+mc)}\left(\frac{\sigma}{2} \times p\right). \tag{26}$$

This will be recognized as the boost generator in the canonical representation of the Poincaré group [17].

3.2. The configuration space representation: higher-order polarization and position operator

The quantization of the relativistic particle in configuration space, when the spin is absent, parallels the Galilean case. There is no first-order polarization containing $\tilde{X}^L_{(a^0)}$ and $\tilde{X}^L_{(p)}$, and a higher-order polarization must be introduced which includes a higher-order left operator $\tilde{X}^{L\ \mathrm{HO}}_{(a^0)}$. This operator becomes

$$\tilde{X}^{L\ \mathrm{HO}}_{(a^0)} = \tilde{X}^L_{(a^0)} + \frac{\mathrm{i}}{\hbar}\left\{\hat{P}^L_0 - mc\right\} \tag{27}$$

where

$$\hat{P}^L_0 \equiv \sqrt{m^2c^2 - \hbar^2\left(\tilde{X}^L_{(a)}\right)^2}$$

and leads to a Schrödinger-like equation, once the whole polarization $\mathcal{P}^{\mathrm{HO}} = \langle \tilde{X}^{L\ \mathrm{HO}}_{(a^0)}, \tilde{X}^L_{(p)}, \tilde{X}^L_{(\epsilon)}\rangle$ has been applied. We find

$$\Psi = \zeta \mathrm{e}^{-(\mathrm{i}/\hbar)p\cdot a}\Phi(a, a^0) \tag{28}$$

where Φ satisfies

$$i\hbar\frac{\partial\Phi}{\partial a^0} = \left\{\sqrt{m^2c^2 - \hbar^2\frac{\partial^2}{\partial a^2}} - mc\right\}\Phi. \tag{29}$$

In the $c \to \infty$ limit, (29) reproduces (12). A simple redefinition $\Phi = \Phi'\mathrm{e}^{(\mathrm{i}mc/\hbar)a^0}$ of the wavefunction will restore the rest mass energy.

We must note in passing that the infinite-order character of $\tilde{X}^{L}_{(a^0)}{}^{\rm HO}$ is due to the restriction to the upper sheet of the mass hyperboloid and that a second-order operator exists,

$$\tilde{X}^{L}_{(a^0)}{}^{\rm 2nd} = \tilde{X}^{L}_{(a^0)} + \frac{\mathrm{i}\hbar}{2mc}\left[\left(\tilde{X}^{L}_{(a^0)}\right)^2 - \left(\tilde{X}^{L}_{(a)}\right)^2\right] \tag{30}$$

which leads to the Klein–Gordon equation. We can say that the polarization containing (27) gives a highest-weight representation, whereas the one containing (30) gives a representation characterized by a given value of a Casimir operator of the group. This duality is also valid for more general non-compact Lie groups.

The quantization of the relativistic spinning particle has an additional difficulty since neither the $\tilde{X}^{L}_{(a^0)}$ generator nor $\tilde{X}^{L}_{(p)}$ are allowed to enter the polarization. In fact, the commutator $[\tilde{X}^{L}_{(p^i)}, \tilde{X}^{L}_{(p^j)}]$ contains a U(1) term. Then, a higher-order operator $\tilde{X}^{L}_{(\boldsymbol{p})}{}^{\rm HO}$ must also be introduced and we shall see that this is the origin of the position operator in our approach.

A solution to the conditions defining a higher-order polarization is given by the following set of left operators:

$$\mathcal{P}^{\rm HO} = \langle \tilde{X}^{L}_{(a^0)}{}^{\rm HO}, \hat{\boldsymbol{Q}}^{L} \equiv \tilde{X}^{L}_{(\boldsymbol{p})}{}^{\rm HO} = \hat{\boldsymbol{R}}^{L} - \frac{1}{mc(\hat{P}^{L}_0 + mc)}\hat{\boldsymbol{\Sigma}}^{L} \times \tilde{X}^{L}_{(\boldsymbol{a})}\quad \tilde{X}^{L}_{(\epsilon)} - \mathrm{i}\boldsymbol{n} \times \tilde{X}^{L}_{(\epsilon)}\rangle \tag{31}$$

where

$$\hat{\boldsymbol{R}}^{L} \equiv \frac{mc}{2}\left(\hat{P}^{L-1}_0 \tilde{X}^{L}_{(\boldsymbol{p})} + \tilde{X}^{L}_{(\boldsymbol{p})}\hat{P}^{L-1}_0\right) \tag{32}$$

$$\hat{\boldsymbol{\Sigma}}^{L} \equiv \tilde{X}^{L}_{(\epsilon)} + \mathrm{i}\hbar \tilde{X}^{L}_{(\boldsymbol{a})} \times \hat{\boldsymbol{R}}^{L}. \tag{33}$$

The configuration-space wavefunctions are:

$$\Psi = \zeta \mathrm{e}^{-(\mathrm{i}/\hbar)p\cdot\hat{O}}\phi(\boldsymbol{a}, a^0)\chi^{(j)}_{\mu}(\epsilon) \tag{34}$$

where

$$\hat{O}\phi\chi^{(j)}_{\mu} = \left[\boldsymbol{a} - \mathrm{i}\hbar\frac{\nabla \times \hat{\boldsymbol{S}}}{\hat{P}^0(\hat{P}^0 + mc)} - \frac{\hbar^2}{2\hat{P}^{02}}\nabla\right]\phi\chi^{(j)}_{\mu} \tag{35}$$

and $\hat{P}^0\phi\chi^{(j)}_{\mu} = [\sqrt{m^2c^2 - \hbar^2\nabla^2}]\phi\chi^{(j)}_{\mu}$ and $\hat{\boldsymbol{S}} = \hbar\Sigma^{(j)}$. Now we have factorized out the 'function' (a pseudo-differential operator indeed) $\mathrm{e}^{-(\mathrm{i}/\hbar)p\cdot\hat{O}}$. Special attention should be paid to the non-commutativity of the exponential factors, for which the relation $\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{A+B}$ obviously does not hold.

The action of the quantum operators preserves the structure of the wavefunction, allowing us to factorize out the exponential factor:

$$\begin{aligned}
\hat{E}\,\phi\chi^{(j)}_{\mu} &= \left(\hat{P}^0 - mc\right)\phi\chi^{(j)}_{\mu} \\
\hat{\boldsymbol{P}}\,\phi\chi^{(j)}_{\mu} &= -\mathrm{i}\hbar\nabla\,\phi\chi^{(j)}_{\mu} \\
\hat{\boldsymbol{K}}\,\phi\chi^{(j)}_{\mu} &= \left[\boldsymbol{a}\frac{\hat{P}^0}{mc} - a^0\frac{\hat{\boldsymbol{P}}}{mc} + \frac{\hat{\boldsymbol{P}} \times \hat{\boldsymbol{S}}}{mc(\hat{P}^0 + mc)} - \frac{\mathrm{i}\hbar\hat{\boldsymbol{P}}}{mc\hat{P}^0}\right]\phi\chi^{(j)}_{\mu} \\
\hat{\boldsymbol{J}}\,\phi\chi^{(j)}_{\mu} &= \left[\hat{\boldsymbol{S}} + \boldsymbol{a} \times \hat{\boldsymbol{P}}\right]\phi\chi^{(j)}_{\mu}.
\end{aligned} \tag{36}$$

In order to see the role played by $\hat{Q}^L$, $\hat{R}^L$ and $\hat{\Sigma}^L$, we can construct the right version of these operators or, more precisely, $\hat{Q} \equiv i\hbar\hat{Q}^R$, $\hat{R} \equiv i\hbar\hat{R}^R$ and $\hat{\Sigma} \equiv -i\hbar\hat{\Sigma}^R$ and let them act on the wavefunctions (34). We find

$$\begin{aligned}\hat{Q}\,\phi\chi_\mu^{(j)} &= \left[a - \frac{a^0}{\hat{p}^0}\hat{P}\right]\phi\chi_\mu^{(j)}\\ \hat{R}\,\phi\chi_\mu^{(j)} &= \left[\hat{Q} + \frac{\hat{P}\times\hat{S}}{\hat{P}^0(\hat{P}^0+mc)}\right]\phi\chi_\mu^{(j)}\end{aligned} \tag{37}$$

$$\hat{\Sigma}\,\phi\chi_\mu^{(j)} = \left[\hat{J} - \hat{R}\times\hat{P}\right]\phi\chi_\mu^{(j)}\,. \tag{38}$$

To compare with standart results [13, 20] we give the expressions of these operators in the momentum space representation:

$$\begin{aligned}\hat{Q}\Phi_\mu^{(j)} &= i\hbar\left(\frac{\partial}{\partial p} - \frac{p}{2p^{02}}\right)\Phi_\mu^{(j)}\\ \hat{R}\Phi^{(j)} &= \left(\hat{Q} + \frac{\hbar p\times\Sigma^{(j)}}{p^0(p^0+mc)}\right)\Phi^{(j)}\\ \hat{\Sigma}\Phi^{(j)} &= \left(\hat{J} + p\times\hat{R}\right)\Phi^{(j)}\end{aligned} \tag{39}$$

suggesting the interpretation of these operators as the Newton–Wigner, mean position and mean spin operators, respectively [18–20]. A natural interpretation that now arises refers to $\hat{S}$ as the intrinsic spin:

$$\hat{S} = \hat{J} + \hat{P}\times\hat{Q}\,. \tag{40}$$

3.3. *Classical solution manifold*

The classical phase space is parametrized by Noether invariants which are given in this approach by the contraction of the right-invariant vector fields with the 1-form Θ. Among them, only the basic ones, that is, the Noether invariants associated with vector fields outside the characteristic subalgebra (the kernel of the Lie algebra co-cycle), are independent. The remainder can be expressed in terms of the former.

From (17) and (19) we see that

$$\begin{aligned}F_{(a^0)} &\equiv i_{\tilde{X}^R_{(a^0)}}\Theta = -(p^0 - mc) \equiv -(P^0 - mc)\\ F_{(a)} &\equiv i_{\tilde{X}^R_{(a)}}\Theta = p \equiv P\\ F_{(\epsilon)} &\equiv i_{\tilde{X}^R_{(\epsilon)}}\Theta = F_{(\epsilon)}{}^{SU(2)} + a\times p + \frac{(S\times p)\times p}{mc(p^0+mc)} \equiv J - jn\\ F_{(p)} &\equiv i_{\tilde{X}^R_{(p)}}\Theta = \frac{p^0}{mc}\left[-a + \frac{a^0}{p^0}p + \frac{S\times p}{mc(p^0+mc)}\left(1 + \frac{mc}{p^0}\right)\right] \equiv -K\end{aligned} \tag{41}$$

where $S \equiv F_{(\epsilon)}{}^{SU(2)} + jn$, introduced in the appendix, is also a conserved quantity, and the definitions $P^0 \equiv -F_{(a^0)} + mc$, $J \equiv F_{(\epsilon)} + jn$ eliminate the effect of the pseudo-extensions.

Using the notation

$$Q \equiv a - \frac{a^0}{P^0}P - \frac{S\times P}{mc(P^0+mc)} = \frac{mc}{P^0}K + \frac{S\times P}{P^0(P^0+mc)}$$

$$\boldsymbol{R} \equiv \boldsymbol{Q} - \frac{\boldsymbol{S} \times \boldsymbol{P}}{P^0(P^0 + mc)} = \frac{mc}{P^0}\boldsymbol{K} \tag{42}$$
$$\boldsymbol{\Sigma} \equiv \boldsymbol{S} - \frac{\boldsymbol{P} \times (\boldsymbol{S} \times \boldsymbol{P})}{P^0(P^0 + mc)}$$

we can rewrite the Noether invariants associated with rotations as

$$\boldsymbol{J} = \boldsymbol{Q} \times \boldsymbol{P} + \boldsymbol{S} = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{\Sigma}. \tag{43}$$

The last formula shows that we can think of $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ as the classical counterpart of the previously introduced quantum operators $\hat{\boldsymbol{Q}}$, $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{\Sigma}}$.

Also the Poincaré–Cartan form, $\Theta_{\mathrm{PC}} = \Theta - d\zeta/i\zeta$, can be written in a more natural way,

$$\Theta_{\mathrm{PC}} = -\boldsymbol{Q} \cdot d\boldsymbol{P} + \frac{(\boldsymbol{S} \times \boldsymbol{n}) \cdot d\boldsymbol{S}}{(j + \boldsymbol{S} \cdot \boldsymbol{n})} \tag{44}$$

which justifies the assumption that $\boldsymbol{Q}$ ($\hat{\boldsymbol{Q}}$) is the position variable (operator). The orbital part of the form Θ_{PC} here appears in Darboux's coordinates. The spin part also appears in a canonical form on the sphere $\boldsymbol{S}^2 = j^2$.

We note finally that the intrinsic angular momentum $\boldsymbol{S}$ turns out to be the spatial part of the Pauli–Lubanski vector W^μ [17] boosted with parameters $-\boldsymbol{P}$, $\boldsymbol{S} = \boldsymbol{W} - (W_0/(P^0 + mc))\boldsymbol{P}$ and $\boldsymbol{\Sigma} = \boldsymbol{W}/P^0$.

4. Concluding remarks

In this paper we have obtained the irreducible representations of the Galilei and Poinceré groups associated with $m \neq 0$ following a group-theoretical approach, and derived well-known expressions for position and spin operators. We would like to stress that the relevant feature is precisely the power of the higher-order polarization technique, which can be applied in exactly the same way to more general systems where the corresponding expressions are unknown. For instance, it can be used to study the relativistic spinning harmonic oscillator [21].

We must remark, nevertheless, that the higher-order polarization technique involves tedious calculations for which symbolic computation of approximate series is often required. In the present case, the series defining the operators $\tilde{X}^{L\ \mathrm{HO}}_{(a^0)}$ and $\tilde{X}^{L\ \mathrm{HO}}_{(p)}$ have been summed up without special difficulty. For more general systems, even the group law, required as the starting point, has to be computed order by order.

Another drawback of the present approach is the ambiguity in finding solutions to the polarization subalgebra itself. In the case of the relativistic particle, for instance, different polarizations can be found for which the rotations operators $\tilde{X}^L_{(\epsilon)}$ are replaced by higher-order ones. However, equivalent polarizations lead to representations which are related by integral transforms.

Finally, we want to mention that the treatment of the representations associated with massless particles could be made in a similar manner provided that the symmetry group is modified in an appropriate way. The conformal group, or a contraction of it, is more suitable as the starting symmetry for the present approach (see [1]).

Appendix A. Irreducible representations of SU(2)$\tilde{\otimes}$U(1)

As the starting point for the application of the GAQ, we want to derive a parametrization of SU(2) in which a principal bundle structure of SU(2) itself over its quotient by a U(1) subgroup is manifest. This U(1) subgroup should not be confused with the U(1) group by which SU(2) is later centrally (pseudo-)extended.

To obtain such a parametrization, we shall write the matrices of SU(2) as $U(\xi) \equiv \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}$ with the condition $\xi^\dagger\xi \equiv |z_1|^2 + |z_2|^2 = 1$. It is apparent that an element of SU(2) is a point on the sphere $S^3 \subset C^2$. The projection $\pi : \xi \to \xi^\dagger\sigma\xi$ goes from S^3 onto the sphere S^2 (σ are the Pauli matrices), as $(\xi^\dagger\sigma\xi)^2 = 1$, and $\pi^{-1}(\xi^\dagger\sigma\xi)$, $\forall\xi$, is isomorphic to the set U(1) of SU(2) matrices with $z_2 = 0$. This U(1) subgroup acts from the right by the usual multiplication U"= U'U, leaving each fibre $\pi^{-1}(\xi^\dagger\sigma\xi)$ invariant.

Locally we take a chart for the base manifold S^2 with parameters z, z^*, and for the fibre we take $\theta = -\mathrm{i}\log\eta$, $\eta \in U(1) \subset SU(2)$. These parameters are related to the non-minimal parameters z_1, z_2 through

$$z_1 = \sqrt{\frac{1+\kappa}{2}}\eta \qquad z_2 = \sqrt{\frac{2}{1+\kappa}}z\eta \qquad \kappa = \sqrt{1-2zz^*}. \tag{A1}$$

(A1) translates the matrix multiplication law into a group law

$$\begin{aligned}
z'' &= z'\eta^{-2} + \kappa' z - \frac{z}{1+\kappa}[z^*z'\eta^{-2} + z^{*\prime}z\eta^2] \\
z^{*\prime\prime} &= z^{*\prime}\eta^2 + \kappa' z^* - \frac{z^*}{1+\kappa}[zz^{*\prime}\eta^2 + z'z^*\eta^{-2}] \\
\eta'' &= \sqrt{\frac{2}{1+\kappa''}}\left\{\sqrt{\frac{1+\kappa'}{2}}\sqrt{\frac{1+\kappa}{2}}\eta'\eta - \frac{1}{2}\sqrt{\frac{1}{1+\kappa'}}\sqrt{\frac{1}{1+\kappa}}(z^*z'\eta^*\eta' + zz^{*\prime}\eta\eta^{*\prime})\right\} \\
\kappa'' &= \kappa'\kappa - (z'z^*\eta^{-2} + z^{*\prime}z\eta^2).
\end{aligned} \tag{A2}$$

The group law for SU(2) in (A2),(A3) is now centrally pseudo-extended by means of the co-boundary generated by the function $e^{2\mathrm{i}j\theta} = \eta^{2j}$. The central part of the group law for SU(2)$\tilde{\otimes}$U(1) is thus

$$\zeta'' = \zeta'\zeta(\eta''\eta'^{-1}\eta^{-1})^{2j}. \tag{A3}$$

From the group law in (A2), (A3) we now obtain the right-invariant vector fields

$$\begin{aligned}
\tilde{X}^R_{(z)} &= \frac{e^{-2\mathrm{i}\theta}}{2(1+\kappa)}\left[(1+\kappa)^2\frac{\partial}{\partial z} - 2z^{*2}\frac{\partial}{\partial z^*} + iz^*\frac{\partial}{\partial\theta} - 2\mathrm{i}jz^*\Xi\right] \\
\tilde{X}^R_{(z^*)} &= \frac{e^{2\mathrm{i}\theta}}{2(1+\kappa)}\left[(1+\kappa)^2\frac{\partial}{\partial z^*} - 2z^2\frac{\partial}{\partial z} - \mathrm{i}z\frac{\partial}{\partial\theta} + 2\mathrm{i}jz\Xi\right] \\
\tilde{X}^R_{(\eta)} &= \frac{\partial}{\partial\theta} \qquad \tilde{X}^R_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi
\end{aligned} \tag{A4}$$

and the left-invariant vector fields

$$\begin{aligned}
\tilde{X}^L_{(z)} &= \kappa\frac{\partial}{\partial z} - \frac{\mathrm{i}z^*}{2(1+\kappa)}\frac{\partial}{\partial\theta} + \frac{\mathrm{i}jz^*}{1+\kappa}\Xi \\
\tilde{X}^L_{(z^*)} &= \kappa\frac{\partial}{\partial z^*} + \frac{\mathrm{i}z}{2(1+\kappa)}\frac{\partial}{\partial\theta} - \frac{\mathrm{i}jz}{1+\kappa}\Xi \\
\tilde{X}^L_{(\eta)} &= \kappa\frac{\partial}{\partial\theta} - 2iz\frac{\partial}{\partial z} + 2iz^*\frac{\partial}{\partial z^*} \qquad \tilde{X}^L_{(\zeta)} = \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi.
\end{aligned} \tag{A5}$$

The right fields in (A4) fulfil the following commutation relations:

$$
\begin{aligned}
[\tilde{X}^R_{(\theta)}, \tilde{X}^R_{(z)}] &= -2\mathrm{i}\tilde{X}^R_{(z)} \\
[\tilde{X}^R_{(\theta)}, \tilde{X}^R_{(z^*)}] &= +2\mathrm{i}\tilde{X}^R_{(z^*)} \\
[\tilde{X}^R_{(z)}, \tilde{X}^R_{(z^*)}] &= -\frac{\mathrm{i}}{2}\tilde{X}^R_{(\theta)} + i\Xi.
\end{aligned}
\tag{A6}
$$

From (A5) one derives the quantization form

$$
\Theta = \frac{\mathrm{i}j}{1+\kappa}(z^* dz - z dz^*) + 2j(\kappa - 1)d\theta + \frac{d\zeta}{\mathrm{i}\zeta} \tag{A7}
$$

and then, using the right-invariant vector fields, one gets the following Noether invariants:

$$
\begin{aligned}
F_{(\theta)} &\equiv i_{\tilde{X}^R_{(\theta)}}\Theta = 2j(1-\kappa) \\
F_{(z)} &\equiv i_{\tilde{X}^R_{(z)}}\Theta = ijz^* \\
F_{(z^*)} &\equiv i_{\tilde{X}^R_{(z^*)}}\Theta = -ijz.
\end{aligned}
\tag{A8}
$$

The pseudo-extension of the group law of SU(2) in (A2), which yielded the group law (A2), (A3) of SU(2)$\tilde{\otimes}$U(1), has turned SU(2) into a true dynamical system where the variable which plays the role of 'time' is θ (see [16] for the related case of SL(2,$\Re$)$\tilde{\otimes}$U(1)). The characteristic module is precisely given by the left-invariant vector field associated with this 'time', that is, $\mathcal{G}_\Theta = \langle \tilde{X}^L_{(\theta)} \rangle$. Furthermore, the coordinates (z, z^*) play the role of a coordinate–momentum pair for this dynamical system.

A look at (A4) shows that we have a full polarization for SU(2)$\tilde{\otimes}$U(1), which is to include the characteristic module and 'half' of the canonically conjugated coordinates. We shall take the polarization as

$$
\mathcal{P} = \langle \tilde{X}^L_{(\theta)}, \tilde{X}^L_{(z)} \rangle. \tag{A9}
$$

The equivariance condition $\Xi\Psi^{(j)} = \mathrm{i}\Psi^{(j)}$ factors the ζ-dependence: $\Psi^{(j)} = \zeta\Phi^{(j)}(z, z^*, \theta)$. The most general solution to the rest of polarization conditions, $\tilde{X}^L_{(\theta)}\Psi^{(j)} = \tilde{X}^L_{(z)}\Psi^{(j)} = 0$, is a combination of functions of the form

$$
\Phi^{(j)}_\nu(z, z^*, \theta) = (1+\kappa)^j \left[\mathrm{e}^{-2\mathrm{i}\theta}(1+\kappa)^{-1}z^*\right]^\nu \tag{A10}
$$

where ν ranges from 0 to $2j$. Writing $\mu = \nu - j$ we get

$$
\Phi^{(j)}_\mu(z, z^*, \theta) = (1+\kappa)^{-\mu}\mathrm{e}^{-2\mathrm{i}(\mu+j)\theta} z^{*\mu+j} \tag{A11}
$$

and the explicit form of the representation given by the action of the operators (right fields in (A4)) on wavefunctions $\Psi^{(j)}_\mu \equiv \zeta\Phi^{(j)}_\mu$ looks like

$$
\begin{aligned}
\tilde{X}^R_{(\theta)}\Psi^{(j)}_\mu &= -2\mathrm{i}(\mu + j)\Psi^{(j)}_\mu \\
\tilde{X}^R_{(z)}\Psi^{(j)}_\mu &= -(j-\mu)\Psi^{(j)}_{\mu+1} \\
\tilde{X}^R_{(z^*)}\Psi^{(j)}_\mu &= \frac{1}{2}(j+\mu)\Psi^{(j)}_{\mu-1}.
\end{aligned}
\tag{A12}
$$

The normalized wavefunctions $\chi^{(j)}_\mu$ are given by

$$
\chi^{(j)}_\mu = (-1)^{\frac{j+\mu}{2}} \prod_{s=1}^{j+\mu} \sqrt{\frac{2(j-\mu+s)}{j+\mu-s+1}}\Psi^{(j)}_\mu \tag{A13}
$$

and defining the operators

$$\hat{J}_+ = -\sqrt{2}\mathrm{i}\tilde{X}^R_{(z)}$$
$$\hat{J}_- = -\sqrt{2}\mathrm{i}\tilde{X}^R_{(z^*)} \qquad \text{(A14)}$$
$$\hat{J}_0 = \frac{\mathrm{i}}{2}\left(\tilde{X}^R_{(\theta)} + 2j\,\Xi\right)$$

(note the constant term in $\hat{J}_0$, which eliminates the effect of the pseudo-extension), we get the typical representation

$$\hat{J}_+\chi^{(j)}_\mu = \sqrt{j(j+1)-\mu(\mu+1)}\chi^{(j)}_{\mu+1}$$
$$\hat{J}_-\chi^{(j)}_\mu = \sqrt{j(j+1)-\mu(\mu-1)}\chi^{(j)}_{\mu-1} \qquad \text{(A15)}$$
$$\hat{J}_0\chi^{(j)}_\mu = \mu\chi^{(j)}_\mu .$$

We wish to discuss briefly the question of the globality of results such as (A10). To be more precise, we may wonder to what extent the solutions to certain differential equations, which have been written in local coordinates, have a global validity. In this example the answer is clear since we can use the "global coordinates" z_1, z_2 and rewrite (A10) in terms of them. Thus, for instance,

$$\Phi^{(j)}_\nu = 2^{j-\nu/2}\mathrm{e}^{2ij\theta}z_1^{j-\nu}z_2^{*\nu} \qquad \text{(A16)}$$

is analytic only for $2j, \nu \in N$ In the general case we must resort to a Fock-like construction of the space of states. Starting from just one globally defined function, the vacuum $|0\rangle$, the states obtained from it by the action of the creation operators (which are global quantities) are global.

In the rest of this appendix we are interested in expressing the previous results on SU(2)⊗U(1) in terms of the coordinates ϵ which were used in [1] as well as in the main text. The change of coordinates is given by

$$\mathrm{U}(\epsilon) = \sqrt{1-\epsilon^2/4} + \frac{\mathrm{i}}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon} \qquad \text{(A17)}$$

(where $\boldsymbol{\sigma}$ are the Pauli matrices and $\epsilon^2 = \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}$), or more explicitly

$$z_1 = \sqrt{\frac{1+\kappa}{2}}\mathrm{e}^{\mathrm{i}\theta} = \sqrt{1-\epsilon^2/4} + \mathrm{i}\epsilon_3/2 \qquad \text{(A18)}$$
$$z_2 = \sqrt{\frac{2}{1+\kappa}}z\mathrm{e}^{\mathrm{i}\theta} = (-\epsilon_2 + \mathrm{i}\epsilon_1)/2.$$

The change in (A18) translates (A2) into the group law

$$\boldsymbol{\epsilon}'' = \sqrt{1-\epsilon'^2/4}\,\boldsymbol{\epsilon} + \sqrt{1-\epsilon^2/4}\,\boldsymbol{\epsilon}' + \frac{1}{2}\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon} \qquad \text{(A19)}$$

which is the same that can be found in (A.4) of [1], setting $\alpha = 0$ and $\epsilon \to -\epsilon/2$. In addition, we shall proceed in a covariant way so that the formulae concerning the Poincaré group can be dealt with more easily. To this end we write the function η that generates the co-boundary as

$$\eta = \frac{\sqrt{1-\epsilon^2/4} + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\epsilon}/2}{\sqrt{1-r^2/4}} \qquad \text{(A20)}$$

where $r^2 = (\epsilon \times n)^2$ has been defined and n is an arbitrary, yet constant, unit vector which afterwards will indicate the spin quantization axis.

From the group law in (A19), and (A3) with $\eta = e^{i\theta}$ given by (A20), one derives the left- and right-invariant vector fields

$$\tilde{X}^L_{(\epsilon)} = \sqrt{1-\epsilon^2/4}\frac{\partial}{\partial\epsilon} - \frac{1}{2}\epsilon \times \frac{\partial}{\partial\epsilon} + \frac{j}{1-r^2/4}\left(\tfrac{1}{2}\sqrt{1-\epsilon^2/4}n \times \epsilon + (\epsilon\cdot n)\epsilon\right)\Xi \tag{A21}$$

$$\tilde{X}^R_{(\epsilon)} = \sqrt{1-\epsilon^2/4}\frac{\partial}{\partial\epsilon} + \frac{1}{2}\epsilon \times \frac{\partial}{\partial\epsilon} - \frac{j}{1-r^2/4}\left(\tfrac{1}{2}\sqrt{1-\epsilon^2/4}n \times \epsilon - (\epsilon\cdot n)\epsilon\right)\Xi \tag{A22}$$

with the commutation relations

$$[\tilde{X}^R_{(\epsilon^i)}, \tilde{X}^R_{(\epsilon^j)}] = -\eta_{ij.}^{\ \ k}\left(\tilde{X}^R_{(\epsilon^k)} + jn_k\Xi\right). \tag{A23}$$

Using the SO(3) element associated with (A17),

$$R(\epsilon) = (1-\epsilon^2/4)I - \sqrt{1-\epsilon^2/4}\epsilon \times \ + \frac{1}{2}(\epsilon\cdot\ )\epsilon \tag{A24}$$

we define the functions $S = jRn$ intimately related to the Noether invariants. They satisfy $S^2 = j^2$. In terms of the new variables the right vector fields, the quantization form and its differential are written as

$$\tilde{X}^R_{(\epsilon)} = S \times \left[n \times \left(n \times \frac{\partial}{\partial S}\right)\right] + \frac{j}{j + S\cdot n} n \times (S \times n)\Xi \tag{A25}$$

$$\Theta = \frac{(n \times S)\cdot \mathrm{d}S}{j + S\cdot n} + \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{A26}$$

$$\mathrm{d}\Theta = \frac{n\cdot(\mathrm{d}S \overset{\times}{\wedge} \mathrm{d}S)}{2n\cdot S}. \tag{A27}$$

It is easy to check that $i_{n\cdot\tilde{X}^L_{(\epsilon)}}\mathrm{d}\Theta = 0$ and that the polarization operators (A9) are both contained in the vectorial expression $\tilde{X}^L_{(\epsilon)} - \mathrm{i}n \times \tilde{X}^L_{(\epsilon)}$.

From the quantization form (A26) we derive the Noether invariants:

$$F_{(\epsilon)} \equiv i_{\tilde{X}^R_{(\epsilon)}}\Theta = S - jn. \tag{A28}$$

The Poisson bracket induced on the sphere $S^2 = j^2$ by $\mathrm{d}\Theta$ reproduces the algebra of SU(2):

$$\left\{S_i, S_j\right\} = \eta_{ij.}^{\ \ k} S_k. \tag{A29}$$

It is worthwile to calculate the action of the operators $\tilde{X}^R_{(\epsilon)}$ on the normalized wavefunctions given in (A13). It is understood that the change of coordinates in (A18) has been performed in the wavefunctions; for instance, the vacuum $|0\rangle$ is written as $(j + n\cdot S)^j$ save for numerical factors. Writing $\chi^{(1/2)}_{-1/2} = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi^{(1/2)}_{1/2} = \begin{pmatrix}0\\1\end{pmatrix}$, we get the matrix form of $\tilde{X}^R_{(\epsilon)}$ for $j = 1/2$:

$$\tilde{X}^R_{(\epsilon)} + jn\,\Xi\Big|_{j=1/2} = \frac{\mathrm{i}}{2}\sigma \tag{A30}$$

where σ are the Pauli matrices.

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